UNIVERSITY OF CALGARY
HASKAYNE SCHOOL OF BUSINESS

Corporate Finance

CAPM (theory)

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Agenda

Capital Asset Pricing Model: the intuition

Portfolio of two assets

- Expected return, variance, correlation and covariance
- Return and volatility of a portfolio of two assets

Portfolio of risky assets

- Variance of a portfolio and the optimal portfolio

Portfolio of risky assets + a risk-free asset

- Risk-free borrowing and lending?

CAPM

- Market equilibrium, beta and the security market line (SML)
- Roll's critique

Practitioners' perspective

Chapter 11 of the textbook

Capital Asset Pricing Model: the intuition

- The future value of securities is uncertain, which is undesirable for risk-adverse investors: diversification is therefore 'valuable' as it is reducing uncertainty.
- If an investment contributes to diversification, this is a 'valuable' characteristic.
- Higher 'value' results in a **higher price** for a security... and therefore a **lower expected return**.
- Assign $\beta = 1$ to the risk contribution of the market portfolio to itself.
- A security having the same risk contribution than the market portfolio to itself ($\boldsymbol{\beta = 1}$) is expected to deliver the **same return as the market**.
- A security having a higher risk contribution than the market portfolio to itself ($\boldsymbol{\beta > 1}$) is expected to deliver a **higher return than the market**.
    - Identical cash flows discounted at a higher rate implies a lower valuation and a lower market price.
- A security having a lower risk contribution than the market portfolio to itself ($\boldsymbol{\beta < 1}$) is expected to deliver a **lower return than the market**.
    - Identical cash flows discounted at a lower rate implies a higher valuation and a higher market price.
- Said otherwise, risk-adverse investors are willing to accept a lower return for a security with a positive expected payoff when the market is down.

Expected return, variance, covariance and correlation of assets a & b 4/21

Outcome	Probability	Ra	Rb	Ra-ERa	Rb-ERb	(Ra-ERa)^2	(Rb-ERb)^2	(Ra-ERa)(Rb-ERb)
Depression	25%	-20%	5%	-37.5%	-0.5%	0.140625	0.000025	0.001875
Recession	25%	10%	20%	-7.5%	14.5%	0.005625	0.021025	-0.010875
Normal	25%	30%	-12%	12.5%	-17.5%	0.015625	0.030625	-0.021875
Boom	25%	50%	9%	32.5%	3.5%	0.105625	0.001225	0.011375
	100%							
Expected return (**ER**)		17.5%	5.5%					
Variance (VAR)		0.066875	0.013225			0.066875	0.013225	
Standard Deviation (SD)		25.9%	11.5%			25.9%	11.5%	
Covariance	-0.004875							-0.004875
Correlation	-0.163925							-0.163925

$$Cov(X,Y) = \sigma_{XY} = E\left(X - \overline{X}\right)\left(Y - \overline{Y}\right) = \sum_{i=1}^{n} p_i \left(x_i - \overline{X}\right)\left(y_i - \overline{Y}\right)$$

$$Corr(X,Y) = \rho_{XY} = \frac{Cov\left(X,Y\right)}{SD\left(X\right) \times SD\left(Y\right)} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \qquad -1 \leq \rho_{XY} \leq 1$$

Return of a portfolio of two risky assets

Return of a portfolio comprised of two assets:

$$
\begin{aligned}
E\left(w_X X + w_Y Y\right) &= \sum_{i=1}^{n} \left(w_X p_i x_i + w_Y p_i y_i\right) \\
&= w_X \sum_{i=1}^{n} p_i x_i + w_Y \sum_{i=1}^{n} p_i y_i \\
&= w_X EX + w_Y EY \qquad\qquad w_X + w_Y = 1
\end{aligned}
$$

- The (expected) return of a portfolio equals the weighed sum of the (expected) returns of its individual components.
- The relationship between the (expected) return of a portfolio and the (expected) returns of its individual components is **linear**.

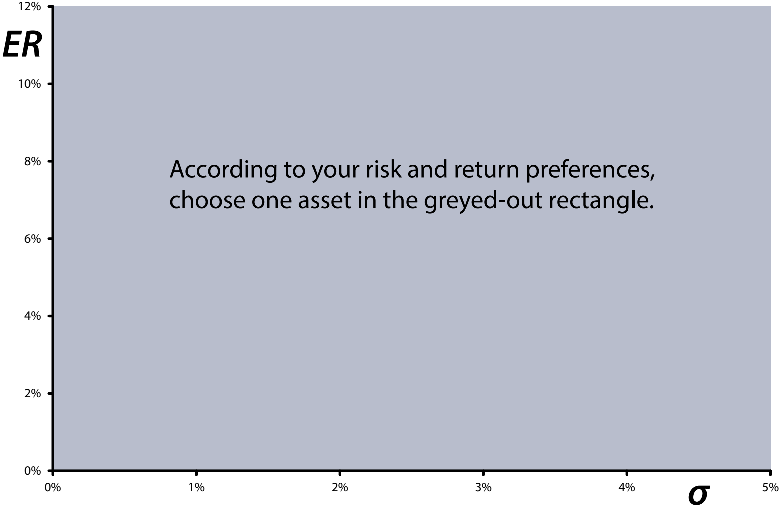
Variance of a portfolio of two risky assets

Variance of a portfolio of two assets using $Var\left(X\right) = E\left(X - EX\right)^2$:

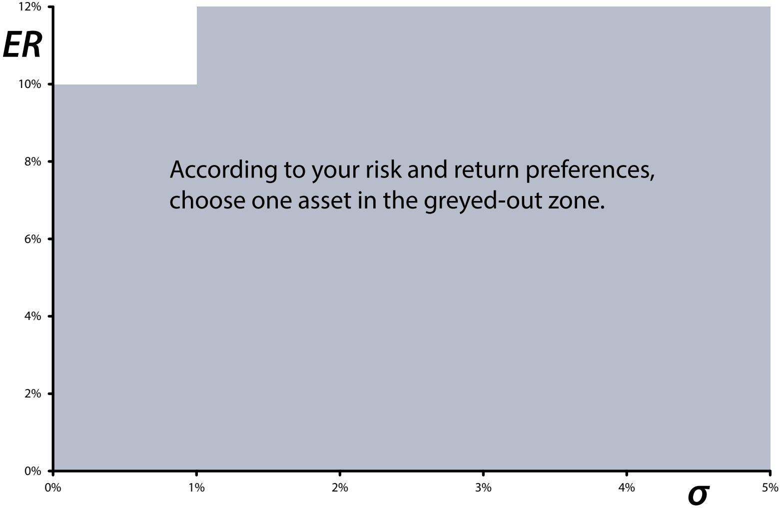
$$
\begin{aligned}
E\left(w_X X + w_Y Y - w_X EX - w_Y EY\right)^2 &= E\left(w_X\left(X - EX\right) + w_Y\left(Y - EY\right)\right)^2 \\
&= E\left(w_X^2\left(X - EX\right)^2 + w_Y^2\left(Y - EY\right)^2 + 2w_X w_Y\left(X - EX\right)\left(Y - EY\right)\right) \\
&= w_X^2 E\left(X - EX\right)^2 + w_Y^2 E\left(Y - EY\right)^2 + 2w_X w_Y E\left(X - EX\right)\left(Y - EY\right) \\
&= w_X^2 Var\left(X\right) + w_Y^2 Var\left(Y\right) + 2w_X w_Y Cov\left(X, Y\right) \\
&= w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_X w_Y \rho_{XY}\sigma_X\sigma_Y
\end{aligned}
$$

- The relationship between the variance of the return of a portfolio and the variance of the returns of its individual components is **not linear**.
- Correlations matter (very good if $\rho_{XY} < 0$ and excellent if $\rho_{XY} = -1$).

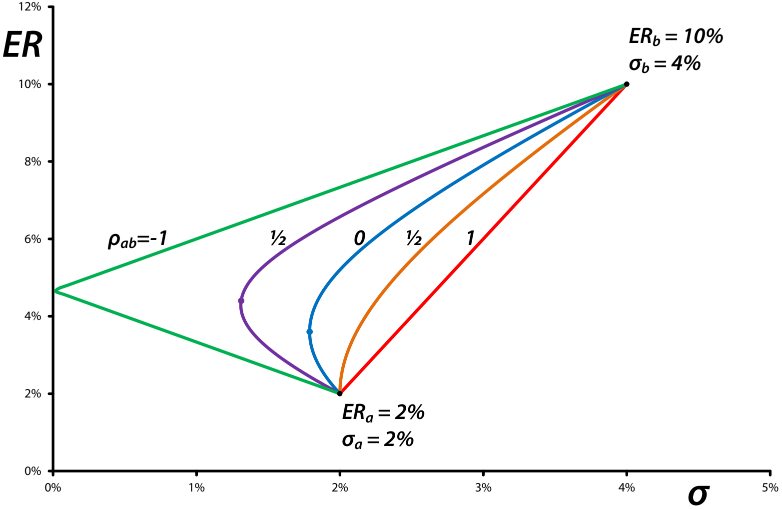
Return and volatility of a portfolio of two risky assets



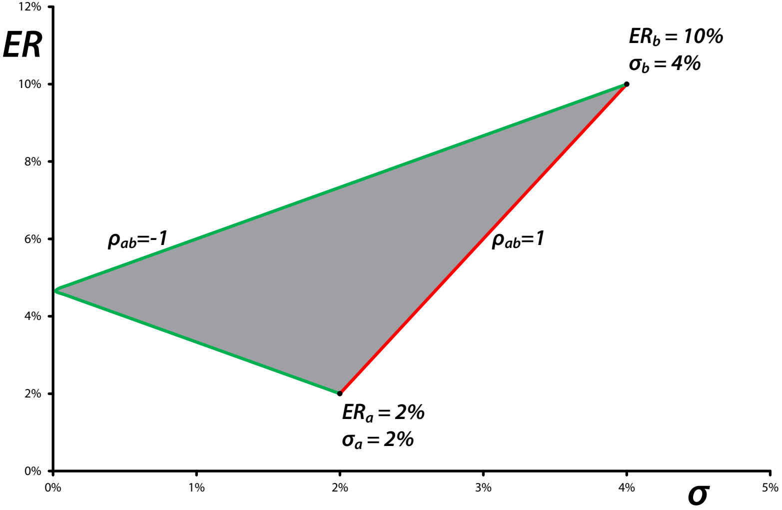
Return and volatility of a portfolio of two risky assets



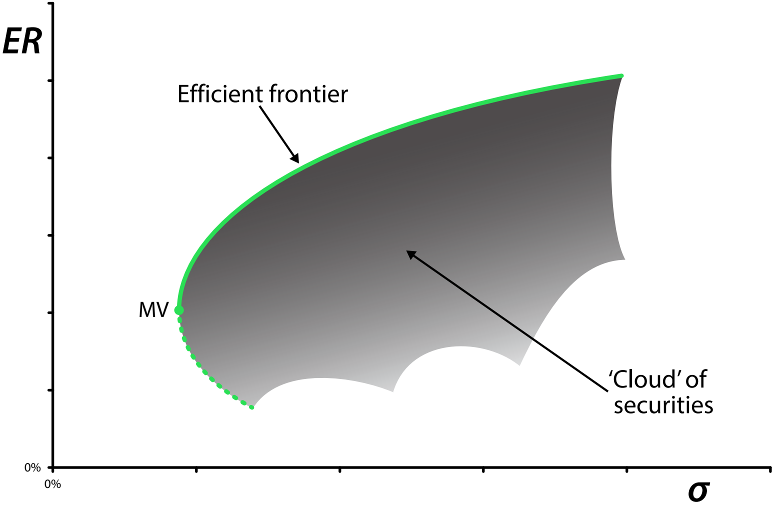
Return and volatility of a portfolio of two risky assets



Return and volatility of a portfolio of two risky assets



Return and volatility of a portfolio of many risky securities



Variance of a portfolio of many risky securities

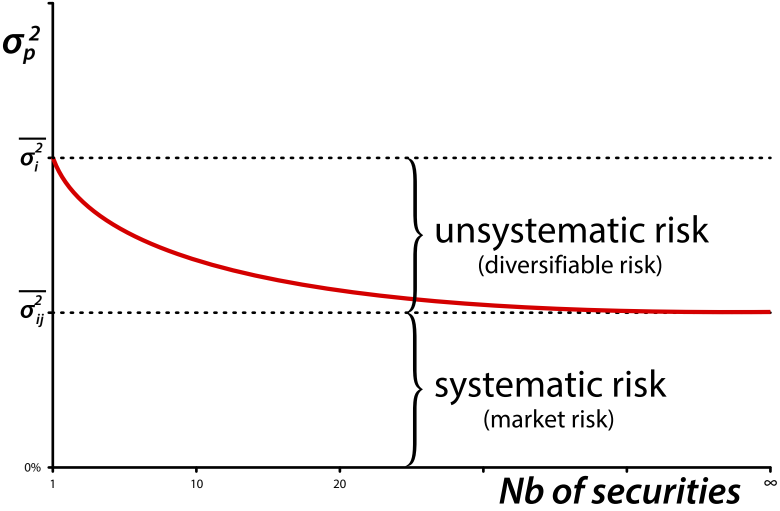
Assumptions for a portfolio of many securities

- All securities have the same variance, denoted $\overline{\sigma_i^2}$.
- All covariances are identical, denoted $\overline{\sigma_{ij}^2}$.
- All securities are equally weighted w_i.

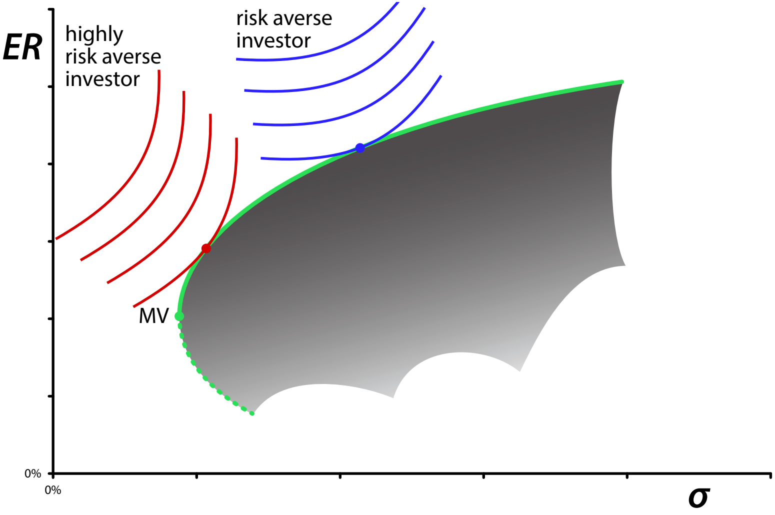
$$\sigma_p^2 = \left(\frac{1}{N}\right)\overline{\sigma_i^2} + \left(1 - \frac{1}{N}\right)\overline{\sigma_{ij}^2} = \overline{\sigma_{ij}^2} \quad \text{as } N \to \infty$$

The variance of such portfolio is **not lower than the average covariance** of the component securities and the variance of each security does not matter much.

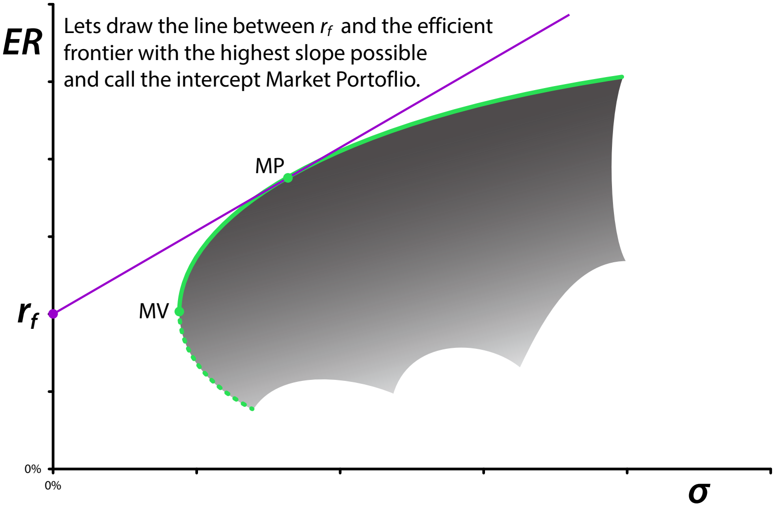
Systematic risk and unsytematic risk



Optimal portfolio of risky assets depends on risk aversion (no Rf)



A risk-free asset allows for risk-free borrowing and lending



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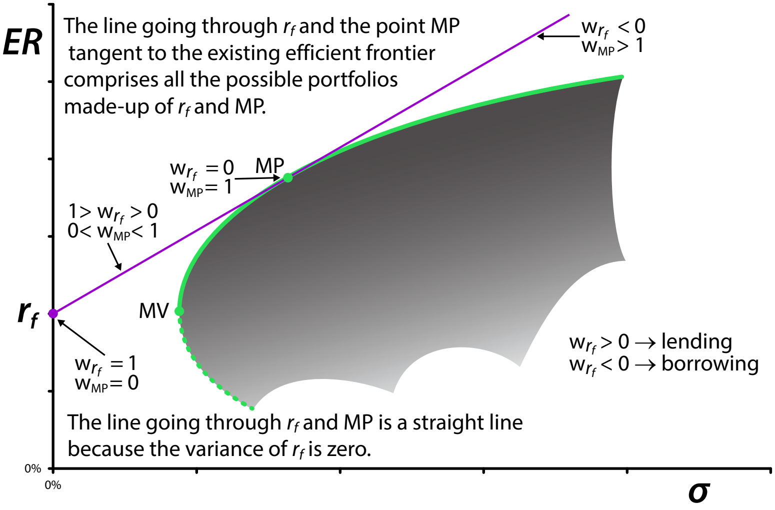

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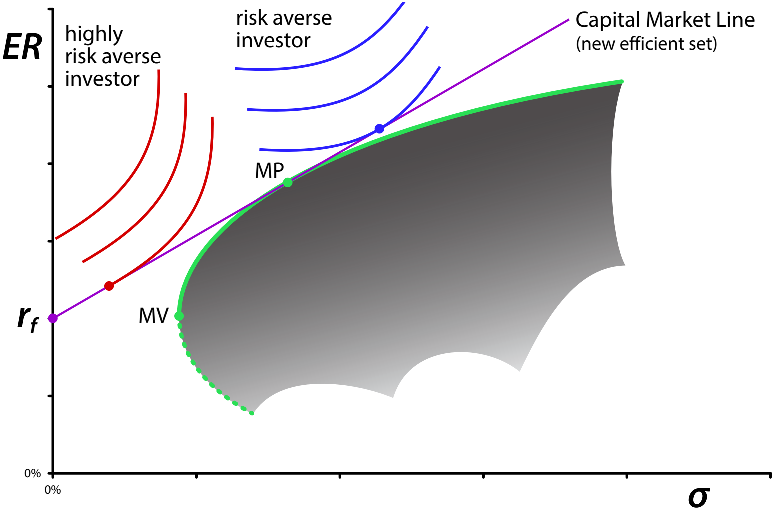


Market equilibrium according to CAPM

The two key assumptions (in addition to the usual technical ones)

- All investors are **mean-variance optimizers** (i.e. only mean and variance matters to them; go for max. return for given volatility).
- All investors have **identical** estimates of expected return, variances, and covariances for every security (i.e. homogeneous expectations).
- All investors therefore have the **same efficient frontier** without R_f.
- With R_f, all investors see as **optimal** to combine R_f and MP.
- Therefore, all investors hold risky securities in the **same proportions** (i.e. in the proportions of the Market Portfolio).

Separation property

- If homogeneous expectations assumptions hold, investors can separate their risk aversion from their choice of portfolio of risky securities!
- The risk aversion of investors is revealed by their individual choice of portfolio along the capital market line.

Beta: measure of exposure/sensitivity to market risk

Beta measures the responsiveness of the return of a security to changes in the return of the market portfolio ($-\infty \leq \beta_i \leq \infty$).

$$\beta_i = \frac{Cov\left(R_i, R_M\right)}{Var\left(R_M\right)} = \frac{\sigma_{R_i,R_M}}{\sigma^2_{R_M}} = \frac{\rho_{R_i,R_M}\sigma_{R_i}\sigma_{R_M}}{\sigma^2_{R_M}} = \rho_{R_i,R_M}\frac{\sigma_{R_i}}{\sigma_{R_M}}$$

β_i quantifies the asset's sensitivity to non-diversifiable risk (sets required expected return).

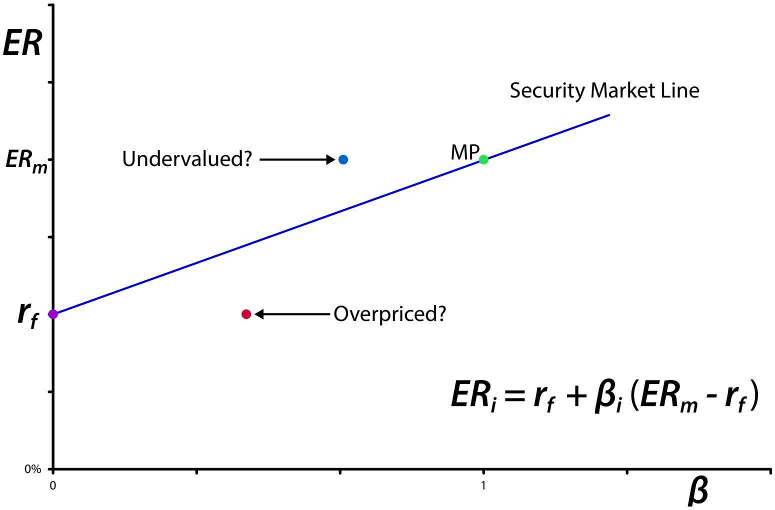
Careful interpretation as β_i is simultaneously influenced by ρ_{R_i,R_M} and σ_{R_i}.

- $\sigma_{R_i} = 0 \rightarrow \beta_i = 0$ (risk-free)
- $\rho_{R_i,R_M} = 0 \rightarrow \beta_i = 0$ (zero systemic risk, but only risk-free if $\sigma_{R_i} = 0$)
- if $\beta_i \neq 0$, its magnitude depends of both ρ_{R_i,R_M} and σ_{R_i}.

The beta for a portfolio is the weighted average of the beta of its components, a linear relationship.

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i$$

Security market line



$$ER_i = r_f + \beta_i\,(ER_m - r_f)$$

Roll's critique

In 1977, Richard Roll got "A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory." published in JFE. The article is now referred to as the 'Roll's critique'.

The article is a well-thought out critique of the CAPM, basically suggesting that the CAPM cannot be satisfactorily empirically tested.

- There is a mean-variance tautology at the core of the model as testing the CAPM is (mathematically) equivalent to testing the mean-variance efficiency of the proxy portfolio used, while the Market Portfolio is assumed to be mean-variance efficient.
- The theory assumes the Market Portfolio to include all assets in existence on the planet. But the returns on many assets classes are unobservable. So the 'true' Market Portfolio is unobservable.

If the 'true' Market Portfolio is unobservable, it is not possible to determine if it is mean-variance efficient (same with any partition as a sub-porfolio), making CAPM not 'truly' testable.

Also, CAPM is a model of expected returns which are also unobservable and proxied using actual/realized returns.

Practitioners' perspective

Survey evidence shows that CAPM is the most common method used by firms to estimate their cost of capital (+ if large, public, CEO has MBA).

- Small firms ask investors, while others also consider historical returns, or use a multi-factor model. Canadian firms seem to lag.

		% of firms using each method					
Survey	Country	Investor required return	Historical stock return	DDM	Some formula	CAPM	Multi Factor
Gitman & Vand. (2000)	U.S.	70	-	14	34	64	1
Graham & Harvey (2003)	U.S.	14	39	16	-	73	34
Baker & Al. (2009)	Canada	20	-	13	52	37	7
Arnold & Hatz. (2000)	U.K. (I.)	-	-	-	-	79	-
Kester & Al. (1999)	Australia	-	-	16	11	73	-

Chapter 11 of the textbook

Textbook sections covered

- 11.1 to 11.10

Worked examples

- 5 worked examples are provided in chapter 11 of the textbook.

Exercises

- 37 exercises are provided in chapter 11 of the textbook.
- You should practice your Excel skills with a few of those.
- Suggest 11.24, 11.25, and 11.34

11.24 Solution

State of Economy	Probability	Return Stock A	Return Stock B	(R - ER)^2 [A]	(R - ER)^2 [B]	(R-ER[A])(R-ER[B])
Bear	33%	10.20%	-4.50%	0.0000284	0.0246490	-0.0008373
Normal	33%	11.50%	14.80%	0.0003361	0.0012960	0.0006600
Bull	33%	7.30%	23.30%	0.0005601	0.0146410	-0.0028637
Expected Return		9.67%	11.20%			
Variance		0.0003082	0.0135287			
Standard Deviation		1.76%	11.63%			
Covariance		-0.001014				
Correlation		-0.496404				

11.25 Solution

State of Economy	Probability	Return Stock A	Return Stock B	(R - ER)^2 [A]	(R - ER)^2 [B]	(R-ER[A])(R-ER[B])
Bear	30%	-2.00%	3.40%	0.0160276	0.0006554	0.0032410
Normal	50%	13.80%	6.20%	0.0009860	0.0000058	0.0000754
Bull	20%	21.80%	9.20%	0.0124100	0.0010498	0.0036094

Expected Return	10.66%	5.96%
Variance	0.0077832	0.0004094
Standard Deviation	8.82%	2.02%
Covariance	0.0017318	
Correlation	0.9701356	

11.34 Solution

State of Economy	Probability	Return Stock 1	Return Stock 1	Return Stock 1	(R - ER)^2 [1]	(R - ER)^2 [2]	(R - ER)^2 [3]	1 & 2 (R-ER[1])(R-ER[2])	2 & 3 (R-ER[2])(R-ER[3])	3 & 1 (R-ER[3])(R-ER[1])
1	15%	20%	20%	5%	0.005625	0.005625	0.005625	0.005625	-0.005625	-0.005625
2	35%	15%	10%	10%	0.000625	0.000625	0.000625	-0.000625	0.000625	-0.000625
3	35%	10%	15%	15%	0.000625	0.000625	0.000625	-0.000625	0.000625	-0.000625
4	15%	5%	5%	20%	0.005625	0.005625	0.005625	0.005625	-0.005625	-0.005625
Expected Return		12.50%	12.50%	12.50%			**Covariance**	0.00125	-0.00125	-0.002125
Variance		0.002125	0.002125	0.002125			**Correlation**	0.5882	-0.5882	-1.0000
Standard Deviation		4.61%	4.61%	4.61%						

50% 1 & 50% 2	
Expected Return	12.5%
Variance	0.0016875
Standard Deviation	4.11%
50% 2 & 50% 3	
Expected Return	12.5%
Variance	0.0004375
Standard Deviation	2.09%
50% 1 & 50% 3	
Expected Return	12.5%
Variance	0
Standard Deviation	0.00%